(9) $\arctan(2x) = 2x - \frac{(2x)^3}{3} + \frac{(2x)^5}{5} \dots$

$$= \sum \frac{(-1)^k (2x)^{2k+1}}{2k+1}.$$

$$\lim_{k\to\infty} \frac{|2x|^{2k+3}}{2k+3} \cdot \frac{2k+1}{|2x|^{2k+1}} = |2x|^2 < 1$$

$$|2x| < 1$$

$$-1 < 2x < 1$$

$$-1/2 < x < 1/2 \qquad R = 1/2$$

(10) $\cos(x^2) = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \dots$

$$= 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{x^{4k}}{(2k)!}$$

$$\lim_{k\to\infty} \frac{|x|^{\cancel{4k}+4}}{(2k+2)!} \cdot \frac{(2k)!}{\cancel{|x|^{4k}}} = \lim_{k\to\infty} \frac{|x|^4 \cancel{(2k)!}}{(2k+2)(2k+1)\cancel{(2k)!}}$$

$$= 0 < 1 \text{ for all } x.$$

So $R = \infty$

(11) $x^2 \sin x = x^2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \dots\right)$

$$= x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \dots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+3}}{(2k+1)!} \qquad R = \infty$$

(similar as above in 10)

12 $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

$\ln(1+\sqrt[3]{x}) = \sqrt[3]{x} - \frac{(\sqrt[3]{x})^3}{2} + \frac{(\sqrt[3]{x})^3}{3} - \cdots$

$= x^{1/3} - \frac{x^{2/3}}{2} + \frac{x^{3/3}}{3} - \cdots$

$= \sum (-1)^{k+1} \frac{x^{k/3}}{k}$

$\lim_{k\to\infty} \frac{|x|^{k+1/3}}{k+1} \cdot \frac{k}{|x|^{k/3}} = |x|^{1/3} < 1$

$|x| < 1$

$-1 < x < 1$

$R = 1$

17 $\frac{1}{2+x} = \frac{1}{2}\left(\frac{1}{1+\frac{x}{2}}\right) = \frac{1}{2}\left(1 - \frac{x}{2} + \left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^3 + \cdots\right)$

$= \frac{1}{2} \cdot \sum (-1)^k \left(\frac{x}{2}\right)^k$

$= \sum \frac{(-1)^k}{2}\left(\frac{x}{2}\right)^k$

$\lim_{k\to\infty} \frac{|x/2|^{\cancel{k+1}}}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{|x/2|^k}} = \frac{|x|}{2} < 1$ $\quad |x| < 2$

$\boxed{R = 2}$

19 $\sin x + \cos x = \sum_{k=0}^{\infty} (-1)^k \left(\frac{x^{2k}}{(2k)!} + \frac{x^{2k+1}}{(2k+1)!}\right)$ $\quad R = \infty$

$\left[1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \cdots = \sin x + \cos x \right.$ Converges for all $\left. x \right]$

22 $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

$$= 2\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} \qquad \text{So } R = 1$$

32 $$\frac{\arctan x}{x} = \frac{1}{x}\left(\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}\right)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2k+1}$$

$$\lim_{x\to 0} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2k+1} = 1.$$

35 Let $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ if $|x|<1$.

$$f'(x) = \frac{1}{(1-x)^2} = \sum n x^{n-1} = \frac{1}{x}\sum n x^n \quad \text{if } |x|<1,\ x\neq 0.$$

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = (1/2)\, f'(1/2) = 2.$$

36 $$\frac{1}{x-1} = \frac{x}{x-1} - 1 = \frac{1}{1-\frac{1}{x}} - 1 = \sum_{k=0}^{\infty}\left(\frac{1}{x}\right)^k - 1$$

$$= \sum_{k=1}^{\infty} \frac{1}{x^k} \quad \text{since } \left|\frac{1}{x}\right| < 1 \text{ when } |x|>1.$$

37 $$\int \frac{1}{1-x} = \int \sum_{k=0}^{\infty} x^k$$

$$-\ln|1-x| = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

The series converges on the interval $[-1, 1]$.
(Apply the ratio test, check the endpoints separately)

45 $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{(2x)^k}{k!}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$\ln(1+x^3) = x^3 - \frac{(x^3)^2}{2} + \frac{(x^3)^3}{3} - \dots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{3k}}{k}$$

$$e^{2x} \ln(1+x^3) = \left(1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \dots\right)\left(x^3 - \frac{(x^3)^2}{2} + \frac{(x^3)^3}{3} - \dots\right)$$

$$= \left(x^3 - \frac{x^6}{\underline{2}_4} + \frac{x^9}{3} - \dots\right) + \left(2x^4 - \frac{2x^7}{2} + \frac{2x^{10}}{3} \dots\right) + \left(\frac{(2x)^2}{2!} x^3 - \frac{(2x)^2}{2!}\frac{x^6}{2} + \dots\right)$$

$$+ \left(\frac{(2x)^3}{\underline{3!}_4} x^3 + \dots\right) + \dots$$

You have to list enough terms to write the terms in increasing degree order. Ordering the terms we will get

$$= x^3 + 2x^4 + 2x^5 + \left(\frac{8x^6}{6} - \frac{x^6}{2}\right) + \dots$$

$$= \underbrace{x^3 + 2x^4 + 2x^5 + \frac{5}{6}x^6}_{\text{First 4-terms.}} + \dots$$

47 $f(x) = \frac{e^x}{1-x} = e^x \cdot \frac{1}{1-x}$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$(1 + x + x^2 + x^3 + \dots)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)$$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) + \left(x + x^2 + \frac{x^3}{2!} + \dots\right) + \left(x^2 + x^3 + \dots\right) + \left(x^3 + x^4 + \dots\right)$$

$$= 1 + 2x + \left(\frac{x^2}{2!} + x^2 + x^2\right) + \left(\frac{x^3}{2!} + \frac{x^3}{3!} + x^3 + x^3\right) + \dots = 1 + 2x + \frac{5x^2}{2} + \frac{8}{3}x^3 + \dots$$

51) $\sum_{k=1}^{\infty} k x^{k-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots\cdots$

Recall $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

$$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots$$

Hence $\sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}$

52) $\sum_{k=0}^{\infty} \frac{x^k}{(k+1)!} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots$

Recall $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$

$$\frac{1}{x} \cdot e^x = \underbrace{\frac{1}{x}}_{\text{subtract}} + \underbrace{1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots}_{\text{This portion is the series in question}}$$

Hence

$$\sum \frac{x^k}{(k+1)!} = \frac{e^x}{x} - \frac{1}{x} = \frac{e^x - 1}{x}$$

53) $\sum_{k=1}^{\infty} (-1)^{k+1} x^k = x - x^2 + x^3 - x^4 + \cdots\cdots$

Recall $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots\cdots\cdots$

So $x - x^2 + x^3 - x^4 + \cdots\cdots = 1 - \frac{1}{1+x}$

$$= \frac{1+x-1}{1+x} = \frac{x}{1+x}.$$

54) $$\sum_{k=1}^{\infty} \frac{(2x)^k}{k} = 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{3} + \cdots\cdots$$

Recall $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots\cdots$$

Integrating both sides

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

The series in question is the composition of $-\ln(1-x)$ with $2x$.

hence $$\sum_{k=1}^{\infty} \frac{(2x)^k}{k} = -\ln(1-2x).$$